Topic 4. Power Series.

The main result is that f is analytic at z_0 iff f can be represented as a convergent power series.
This is similar to Taylor Expansion.

1. Definitions

Given a sequence of complex numbers

$$w_0, w_1, \cdots w_n, \cdots$$

If $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} w_i = w$ exists, we say this series converges to w.
or, it's diverge.

One useful tool is to observe a partial sum

$$\sum_{k=m}^{n} w_k \to 0 \quad \text{as} \quad m, n \to \infty.$$

This is called Cauchy Sequence.

Sequence $\sum\limits_{n=0}^{\infty} w_n$ is said to converge absolutely if $\sum\limits_{n=0}^{\infty} |w_n|$ converges. Recall that, in real number world.

$$\sum_{n=0}^{\infty} |w_n| \text{ converges} \iff \sum_{n=0}^{\infty} |w_n| \text{ is bounded.}$$

2. Two useful tools for Convergence absolutely

① Ratio Test.

If $\{w_n\}$ is a seq of non-zero terms,

$$\begin{cases} \limsup\limits_{n\to\infty} \left|\frac{w_{n+1}}{w_n}\right| < 1 \Rightarrow \text{converge absolutely} \\ \liminf\limits_{n\to\infty} \left|\frac{w_{n+1}}{w_n}\right| > 1 \to \text{diverge} \\ \text{o.w} \to \text{don't know} \end{cases}$$

② Root test

$$\begin{cases} \limsup\limits_{n\to\infty} |w_n|^{1/n} < 1 & \text{converge absolutely} \\ \liminf\limits_{n\to\infty} |w_n|^{1/n} > 1 & \text{diverge} \\ \text{o.w} & \text{don't know.} \end{cases}$$

3. Theorem for convergence.

$\{f_n(z)\}$ is a seq of complex-valued function on a set S. ($\mathcal{D} \downarrow S$)
Then $\{f_n(z)\}$ converges point-wise on S (i.e. $\forall z' \in S$, $\{f_n(z')\}$ converges)

iff

$\{f_n(z)\}$ is point-wise Cauchy (i.e. $\forall z' \in S$, $\{f_n(z')\}$ is Cauchy Sequence)

Also $\{f_n(z)\}$ converges unformly

iff

$\{f_n(z)\}$ is uniformly Cauchy on S
i.e. $|f_n(z) - f_m(z)| \to 0$ as $m, n \to \infty$ uniformly for $z \in S$.

4. Weierstrass M-Test.

Let $g_1, g_2 \ldots$ be complex-valued functions on S and assume $|g_n(z)| < M_n$ for $\forall z \in S$. If $\sum\limits_{i=1}^{\infty} M_n < \infty$ then $\{f_n(z)\}$ converge uniformly on S.

5. Power Series.

Consider series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where a_n, z_0 are complex numbers.

If we define $f_n(z) = a_n(z-z_0)^n$, the power series is

$$\sum_{n=0}^{\infty} f_n(z)$$

$f_n(z) = a_n(z-z_0)^n$ is a very simply analytic function.

6. Abel Theorem

If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at point z with $|z-z_0| = r$, then the series converges absolutely on open disk centered at z_0, radius is r. and uniformly converges on each compact set of the open disk

Proof: $\forall z' \in D(z_0, r)$

$$|a_n(z'-z_0)^n| = |a_n(z-z_0)^n| \cdot \left|\frac{z'-z_0}{z-z_0}\right|^n$$

$\because$ the series converges at z

$\therefore |a_n(z-z_0)^n| \xrightarrow{n\to\infty} 0$, or $|a_n(z-z_0)^n| < M$. is bounded

and we know $z' \in D(z_0, r)$

$$\therefore \left|\frac{z'-z_0}{z-z_0}\right|^n \le \frac{|z'-z_0|^n}{|z-z_0|^n} = \left(\frac{r'}{r}\right)^n \le 1 \quad (r' = |z'-z_0|)$$

$\therefore$ $\Sigma a_n (z'-z_0)$ also absolute converges.
From Weierstrass M-Test, it is also uniformly converge

But we only proved the convergence for $|z'-z_0|<|z-z_0|$.
For $|z'-z_0| = |z-z_0|$, we don't know anything.

Corallony:
If a power series diverges at z, than for $\forall z'$ s.t. $|z'-z_0| > |z-z_0|$ diverges.

7. Some Propenties:
If we know $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at z'. the series can heve 3 cases

① diverge everywhere except z'
E.g. $\Sigma n^n (z-z')^n$

② Converge everywhere.
E.g. $\Sigma n^{-n} (z-z')^n$

③ $\exists z_1$ & z_2, s.t

$$\begin{cases} \text{series converges if } |z-z_0| < |z_1-z_0| = R_C \\ \text{series diverges if } |z-z_0| > |z_2-z_0| = R_D. \end{cases}$$

Here, R_c is called convergence radius.

8 Compute Converge Radius for $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l \quad \text{--- D'Alembert}$$

or (equivalent) $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{---- Cauchy}$$

or (equivalent) $$\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = l \quad \text{Cauchy-Hadamard}$$

then, the Converge Radius R_c is

$$R_c = \begin{cases} 1/l & l \neq 0,\ l \neq +\infty \\ 0 & l = \infty \\ +\infty & l = 0 \end{cases}$$

9. Power Series's Properties.

① Power Series analytic at $|z-z_0| < R_c$

② Power Series has infinite order of derivative.

and if denote

$$F(z) = \left(\sum a_n (z-z_0)^n\right)$$

$$F^{(P)}(z) = P!\,C_P + \sum_{i=1}^{\infty} \frac{(P+i)!}{i!} C_{P+i} (z-z_0)^i$$

where $C_P = \dfrac{F^{(P)}(z_0)}{P!}$

③ if we integrate $\sum a_n (z-z_n)$ over a closed path in $|z-z_0| < R_c$, the result is also a power series of same R_c.

i.e. $\sum n C_n z^{n-1} \xleftarrow{\text{derivative}} \sum C_n z^n \xrightarrow{\text{integral}} \sum \frac{C_n}{n+1} z^{n+1}$

have same R_c.

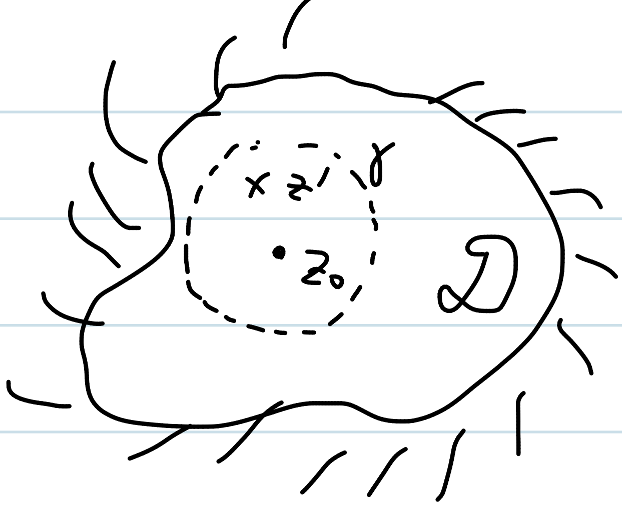
10 Taylor Expansion to Analytic Function

$f(z)$ is analytic in $\mathcal{D}$, $z_0 \in \mathcal{D}$. if $K = \{z : |z-z_0| < R\}$ and $K \subseteq \mathcal{D}$, then $f(z)$ can be expanded uniquely.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(\xi - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!}$$

Proof: ① Correctness



According to Cauchy Integration Formula

$$f(z') = \frac{1}{2\pi i}\int_\gamma \underbrace{\boxed{\frac{f(z)}{z-z'}}}_{(1)} dz$$

$$\frac{f(z)}{z-z'} = \frac{f(z)}{z-z_0}\cdot\frac{z-z_0}{z-z'}$$

$$= \frac{f(z)}{z-z_0}\cdot\frac{1}{1-\frac{z'-z_0}{z-z_0}}$$

Recall that $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \quad |u|<1$

$$\therefore \frac{1}{1-\frac{z'-z_0}{z-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n \leftarrow \text{uniformly converge}$$

and $\frac{f(z)}{z-z_0}$ is bounded. So $\frac{f(z)}{z-z_0}\cdot\frac{1}{1-\frac{z'-z_0}{z-z_0}}$ converge uniformly.

$$\frac{f(z)}{z-z'} = \sum_{n=0}^{\infty}\frac{f(z)}{z-z_0}\cdot\left(\frac{z'-z_0}{z-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{f(z)}{(z-z_0)^{n+1}}\cdot(z'-z_0)^n$$

Plug in the integration

$$f(z') = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z'}dz = \frac{1}{2\pi i}\int \sum_{n=0}^{\infty} \frac{f(z)}{(z-z_0)^{n+1}} \cdot (z'-z_0)^n$$

$$= \sum_{n=0}^{\infty} (z'-z_0)^n \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz}_{= \frac{f^{(n)}(z_0)}{n!}}$$

(TB proved)

$$= \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{a_n} (z'-z_0)^n$$

Prove the integration

One method is using z transformation, it's obvious.

Another method is a general form.

$$f^{(n)}(z') = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z')^{n+1}} dz$$

By deduction

① $n=1$

$$\frac{f(z'+\Delta z) - f(z')}{\Delta z} = \frac{1}{\Delta z}\left[\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z'-\Delta z} dz\right) - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z'} dz\right]$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z'-\Delta z)(z-z')} dz$$

We need to prove that,

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z'-\Delta z)(z-z')}dz\right| \xrightarrow{\Delta z \to 0} \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z')^2}dz\right| \quad --(\divideontimes)$$

Take the difference.

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z'-\Delta z)(z-z')}dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z')^2}dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_\gamma \frac{\Delta z\, f(z)}{(z-z'-\Delta z)(z-z')^2}dz\right|$$

Let $|f(z)| \leq M$ for $z \in \gamma$

and $|z-z'| \geqslant d$

if we let $|\Delta z| < \frac{d}{2}$, then

$$|z-z'-\Delta z| \geqslant |z-z'| - |\Delta z| > d - \frac{d}{2} = \frac{d}{2}$$

$$\therefore \left|\int_\gamma \frac{\Delta z\, f(z)}{(z-z'-\Delta z)(z-z')^2}dz\right| \leq \left|\int_\gamma \frac{\Delta z M}{\frac{d}{2}\cdot d^2}dz\right|$$

if length of γ is l, then

$$\left|\int_\gamma \frac{\Delta z M}{\frac{d}{2}d^2}dz\right| = \frac{Ml}{\frac{d^3}{2}}\Delta z$$

$\therefore \forall \varepsilon > 0$, let $|\Delta z| < \min\left(\frac{d}{2}, \frac{Ml}{\frac{d^3}{2}}\varepsilon\right)$, we can make the difference smaller than ε.

So, $(\divideontimes)$ proved.

we can repeat this for any n. ✓

② Prove uniqueness.

If $f(z) = \sum_{n=0}^{\infty} a_n' (z - z_0)^n$

from derivative to n-th order, we know

$$a_n' = \frac{f^{(n)}(z_0)}{n!} \underset{\uparrow}{=} a_n$$

uniquely equal. ✓

11. Useful Corollaries

① If f is analytic on Ω. f has all orders of derivatives on Ω.

Proof. We have proved in previous page that

$$f^{(n)}(z') = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z')^{n+1}} dz \quad z' \in \Omega$$

② If f has a primitive on Ω, then f is analytic on Ω.

Proof: let F be one of the primitives, use ①

$F^{(1)}$ is also analytic

③ If f continuous on Ω and analytic on $\Omega / \{z_0\} \rightarrow f$ analytic on Ω.

Proof Choose a disk $D \subseteq \Omega$, and f has primitive on D from ②, and it is analytic on D. from ②. Then $\because$ D is arbitrary, $\therefore$ f analytic on Ω.

12. Morera's Theorem

f is continuous on Ω. $\int_T f(z)\,dz = 0$ where T is $\forall$ triangle that $\operatorname{conv}(T) \subseteq \Omega$, then f is analytic on Ω.

E.g. Compute Taylor Expansions.

We can use formulas to compute the coeffients of each term, but its better to use known expansions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad |x| < \infty$$

So,
$$\cos z = \frac{\cos^{iz} + e^{-iz}}{2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$$